Ljusternik-Schnirelmann Categories, Links and Relations

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Abstract

This paper is concerned with some well-known Ljusternik-Schnirelmann categories. We desire to find some links and relations among them. This has been done by using the concepts of precategoty, T-collection and closure of a category.

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1 Introduction

In Ljusternik-Schnirelmann theory, there are several categories with different properties. First of all, Ljusternik introduced a category by means of closed contractible sets which is known as (classical) Ljusternik-Schnirelmann category [4]. Then Schnirelmann estimated this category by using cohomology theory. This method is still the most convenient way to compute LS categories. For this reason, some authors use the same notations for cohomological category [6]. The cuplength category is also derived from this approach to LS theory. Besides these categories, there is another LS category which is similar to the classical one and it is defined by means of open contractible subsets [3]. For the excision property of open subsets, this category is preferable especially in computation.

In the literature, all of the above categories are known as Ljusternik-Schnirelmann category and it is customary to use the notation *cat* for them. In order to distinguish

these categories, we call them: (classical) Ljusternik-Schnirelmann (shortly LS) category, Cohomology Ljusternik-Schnirelmann (CLS) category, Cup Length (CL) category and Homotopy Ljusternik-Schnirelmann (HLS) category, respectively. The aim of this paper is to find some links and relations among these categories.

First of all in section 2, we define the concepts of category and precategory. Indeed, both LS and CL categories are precategories (but not necessarily categories) in an arbitrary topological space. In section 3, we compare LS and HLS categories and find a relation between them. In section 4, we introduce the concept of T-collection which gives an extension of HLS and CLS categories. Finally in section 5, we prove that the cuplength map is a precategory and hence defines a category. This category has been used in [1] to prove the Arnold conjecture for even dimensional tori. We also find a relation between CL and CLS categories in this section.

2 Category

Let M be a topological space. A category on M is a map $\nu : 2^M \longrightarrow \mathbb{Z} \cup \{+\infty\}$ which satisfies the following axioms:

i) If $A \subset B$, then $\nu(A) \leq \nu(B)$.

ii) $\nu(A \cup B) \leq \nu(A) + \nu(B)$.

iii) For every subset $A \subset M$ there exists an open set $U \subset M$ such that $A \subset U$ and $\nu(A) = \nu(U)$.

iv) If $f : M \longrightarrow M$ is a continuous map homotopic to the identity id_M, then $\nu(A) \leq \nu(f(A))$ for every subset $A \subset M$.

It has been shown that if φ^t is a continuous flow on a compact metric space M with a Morse decomposition $M_1, \cdots, M_n$, then $\nu(M) \leq \sum_{i=1}^{n} \nu(M_i)$ for every category ν on M. This result leads to a critical point theory for gradient-like flows. (See [1] and [8] for details.)

Example 1. (Ljusternik-Schnirelmann category). A metric space X is an absolute neighborhood extensor, shortly an ANE, if for every metric space Y, every closed subset

B of Y and $f : B \longrightarrow X$ continuous, there exists a continuous extension of f defined on a neighborhood of B in Y. For a subset $A \subset X$, we define $\nu_{LS}(A)$ to be the minimum number of closed sets contractible in M required to cover A. When X is an ANE, ν_{LS} is a category on X (See [7], § 4.6.) which satisfies the following axiom:

v) If A consists of a single point, then $\nu(A) = 1$.

Note that ν_{LS} is defined on every topological space and satisfies axioms (i), (ii), (iv) and (v). But there are examples of compact metric spaces in which ν_{LS} does not satisfy axiom (iii).

Example 2. (Precategory) Let X be a topological space and τ be the topology on X. A precategory is a map $\nu_0 : \tau \longrightarrow \mathbb{Z} \cup \{+\infty\}$ which satisfies axioms (ii) and (iv). For example ν_{LS} defines a precategory on every topological space. In section 5, we shall show that the cuplength map is also a precategory. Given a precategory ν_0 on X, we can define a category $\widetilde{\nu_0}$: $2^M \longrightarrow \mathbb{Z} \cup \{+\infty\}$ as follows:

$$\widetilde{\nu_0}\ (A) = \min\{\nu_0(U) | U \subset X \text{ open, containing } A\}$$

Notice that ν_{LS} defines a category on X if and only if $\nu_{LS} = \widetilde{\nu}_{LS}$. Thus $\widetilde{\nu}_{LS}$ can be considered as a generalization of Ljusternik-Schnirelmann category.

Example 3. (Closure of a category) Let ν be a category on a normal topological space X. We define $\overline{\nu} : 2^M \longrightarrow \mathbb{Z} \cup \{\infty\}$ by $\overline{\nu}(A) = \nu(\overline{A})$. It is not hard to see that $\overline{\nu}$ is a category. (Normality is used to prove axiom (iii)). We have $\nu \leq \overline{\nu}$ and if ν satisfies axiom (v), then so does $\overline{\nu}$.

3 HLS Category

Let M be a topological space. A subset $A \subset M$ is called contractible in M if the inclusion map $A \longrightarrow M$ is homotopic to a constant. We define Homotopy Ljusternik-Schnirelmann category as follows:

Definition. The HLS-category $\nu_H(A) = \nu_H(A, M)$ of a subset $A \subset M$ is defined to

be the minimum number of open sets contractible in M required to cover M. If such a covering does not exist, we set $\nu_H(A) = +\infty$ and if it exists, A is called H-categorizable (in M).

It is easy to see that ν_H satisfies axioms (i)-(iii). Moreover a subset $A \subset M$ is H-categorizable if and only if $\nu_H(\{x\}) = 1$ for every $x \in A$. Thus ν_H satisfies axiom (v) if and only if M is H-categorizable. The following lemma gives a generalization of axiom (iv).

Lemma 3.1. If Y dominates X, i.e. there are continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ with $g \circ f \sim id_X$, then for every H-categorizable subset $A \subset Y$, $f^{-1}(A)$ is H-categorizable and $\nu_H(f^{-1}(A)) \leq \nu_H(A)$. In particular if Y is H-categorizable, then so is X and $\nu_H(X) \leq \nu_H(Y)$. Hence ν_H is an invariant of homotopy type.

Proof. It is enough to prove that for every open set $U \subset Y$ contractible in Y, $f^{-1}(U)$ is contractible in X. Consider the following commutative diagram in which i and j are inclusion maps:

$$\begin{array}{ccc} H^k(X) & \xleftarrow{f^*} & H^k(Y) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle j^*} \\ H^k(A) & \xleftarrow{(f|_A)^*} & H^k(f(A)) \end{array}$$

$$f \circ i = j \circ f|_V \Rightarrow g \circ f \circ i = g \circ j \circ f|_V \Rightarrow i \sim g \circ j \circ f|_V \sim \text{constant.} \ \square$$

The rest of this section concerns a comparison between ν_H and ν_{LS}. We first prove that every ANE is H-categorizable.

Definition. A topological space X is called semi-locally contractible if for every $x \in X$ and open set $U \subset X$ with $x \in U$, there exists a neighborhood V of x such that $x \in V \subset U$ and V is contractible in U.

Proposition 3.2. Every ANE is semi-locally contractible, hence it is H-categorizable.

Proof. Suppose that X is an ANE, $x_0 \in X$ and $U \subset X$ is an open set such that $x_0 \in U$.

We define $f_0 : \overline{U} \times \{0,1\} \cup \{x_0\} \times [0,1] \longrightarrow X$ by $f_0(x,t) = \begin{cases} x & \text{if } t = 0 \\ x_0 & \text{otherwise.} \end{cases}$

f_0 is continuous and $\overline{U} \times \{0,1\} \cup \{x_0\} \times [0,1]$ is a closed subset of $X \times [0,1]$. Since X is an ANE, there exists a continuous extension f_1 of f_0 defined on a neighborhood of $\{x_0\} \times [0,1]$. We have $f_1(x_0,t) = f_0(x_0,t) = x_0 \in U$. Thus $f_1^{-1}(U)$ is an open set containing $\{x_0\} \times [0,1]$. By compactness of $[0,1]$, there is an open set $V \subset U$ such that $V \times [0,1] \subset f_1^{-1}(U)$. Now consider $f = f_1|_{V\times[0,1]} : V \times [0,1] \longrightarrow U$. We have:

$$f(x,0) = f_1(x,0) = f_0(x,0) = x \quad \text{and} \quad f(x,1) = f_1(x,1) = f_0(x,1) = x_0.$$

It means that V is contractible in U. $\square$

Proposition 3.3. If ν_{LS} defines a category on a normal topological space X, then $\nu_{LS} = \overline{\nu}_H$. In particular ν_{LS} and ν_H agree on closed subsets of an *ANE*.

Proof. Step 1. $\nu_{LS}(A) = \nu_{LS}(\overline{A})$ for every subset $A \subset X$.

Since $\nu_{LS}(A) \leq \nu_{LS}(\overline{A})$, we may assume that $\nu_{LS}(A) < \infty$. Let $A \subset \bigcup_{i=1}^{n} A_i$ where each A_i is closed contractible in X. Thus $\overline{A} \subset \bigcup_{i=1}^{n} A_i$ and it follows that $\nu_{LS}(\overline{A}) \leq \nu_{LS}(A)$.

Step 2. $\nu_H(A) \leq \nu_{LS}(A)$ for every subset $A \subset X$.

Suppose that $A \subset \bigcup_{j\in J} A_j$ and each A_j is closed contractible in X. Thus $\nu_{LS}(A_j) = 1$ for every $j \in J$ and by axiom (iii), there exists an open set U_j with $A_j \subset U_j$ and $\nu_{LS}(U_j) = 1$. Now each U_j is open contractible in X and $A \subset \bigcup_{j\in J} U_j$.

Step 3. For every closed subset $A \subset X$, $\nu_{LS}(A) \leq \nu_H(A)$.

We may assume that $\nu_H(A) < \infty$. Suppose that $A \subset \bigcup_{i=1}^{n} U_i$ where each U_i is open contractible in X. We have $A = \bigcup_{i=1}^{n} A \cap U_i$ and each $A \cap U_i$ is open in A. By Shrinking Lemma [5], there are $V_i \subset A$ open in A such that $A = \bigcup_{i=1}^{n} V_i$ and $\overline{V}_i \subset A \cap U_i$. (The closure is taken in A which is a closed subset of X and there is no ambiguity.) Since $\overline{V}_i \subset U_i$ and U_i is contractible in X, each $\overline{V}_i$ is closed contractible in X and $A \subset \bigcup_{i=1}^{n} \overline{V}_i$.

The proof of Proposition 3.3. is easy now. By steps 1 and 2, $\nu_H(\overline{A}) \leq \nu_{LS}(\overline{A}) = \nu_{LS}(A)$

and by steps 1 and 3, $\nu_{LS}(A) = \nu_{LS}(\overline{A}) \leq \nu_H(\overline{A})$. $\square$

the relation $\nu_{LS} = \overline{\nu}_H$ says that ν_{LS} can be obtained by ν_H. Moreover the above two results show that ν_H is applicable to more spaces than ν_{LS}.

4 T-collection

Having a glance at the definition of HLS category, one can easily find a general method to define a category. We do this by using the concept of T-collection. A generalization of this concept has been used in [8] to obtain a result in critical point theory.

Definition. (a) A collection T of open subsets of M is called a T-collection if for every continuous map $f : M \longrightarrow M$ homotopic to id_M, $U \in T$ implies $f^{-1}(U) \in T$. Every $U \in T$ is called T-trivial in M. A subset $A \subset M$ is called T-categorizable if A is covered by some T-trivial subsets.

(b) For every T-collection T on M, we define the associated category $\nu_T(A)$ to be the minimum number of open sets in T required to cover $A \subset M$. If such a covering does not exist, we set $\nu_T(A) = +\infty$. It is not hard to see that ν_T is a category on M.

Example 1. Let $U \subset M$ be an open set and T be the set of all $f^{-1}(U)$ where $f : M \longrightarrow M$ is continuous homotopic to id_M. Then T is a T-collection.

Example 2. If T_j's are T-collections for $j \in J$, then so are $\bigcap_{j\in J} T_j$ and $\bigcup_{j\in J} T_j$. It is easy to see that every T-collection can be obtained by these two examples.

Definition. Suppose that $\nu : 2^M \longrightarrow \mathbb{Z} \cup \{+\infty\}$ satisfies axiom (iv). For every $n \in \mathbb{Z}$ we define $T_{\nu,n} = \{U \subset M | U \text{ is open and } \nu(U) \leq n\}$ which is obviously a T-collection. If ν is a category, then $T_{\nu,1}$ is called the associated precategory and it is denoted by T_ν.

Now there are two natural questions: 1. When a given category is associated to some T-collection? 2. When a given T-collection is associated to some category? The rest of this section concerns these two questions. First of all, notice that by the definition of ν_T, we have $T \subseteq T_{\nu_T}$.

Lemma 4.1. For every category ν on M and $n \in \mathbb{N}$, $\nu \leq n\nu_{T_{v,n}}$. In particular $\nu \leq \nu_{T_\nu}$.

Proof. Let A be any subset of M. We may assume that $\nu_{T\nu}(A) < \infty$ and $A \subset \bigcup_{i=1}^{m} U_i$ where $U_i \in T_{\nu,n}$ and $m = \nu_{T_{\nu,n}}(A)$. Now by axiom (ii), $\nu(A) \leq \sum_{i=1}^{m} \nu(U_i) \leq mn = n\nu_{T_{\nu,n}}(A)$. $\square$

Proposition 4.2. i) $T_{\nu_{T_\nu}} = T_\nu$ for every category ν on M.
ii) $\nu_{T_{\nu_T}} = \nu_T$ for every T-collection T on M.
Proof. Notice that if $\nu \leq \nu'$ then $T_\nu \supseteq T_{\nu'}$ and if $T \subseteq T'$ then $\nu_T \geq \nu_{T'}$. Now we have $T \subseteq T_{\nu_T}$, hence $T_\nu \subseteq T_{\nu_{T_\nu}}$ and $\nu_T \geq \nu_{T_{\nu_T}}$. Similarly $\nu \leq \nu_{T_\nu}$, hence $\nu_{T_{\nu_T}}$ and $T_\nu \supseteq T_{\nu_{T_\nu}}$. $\square$

Corollary 4.3 i) For every category ν on M, $\nu = \nu_{T_\nu}$ if and only if $\nu = \nu_T$ for some T-collection T on M.
ii) For every T-collection T on M, $T = T_{\nu_T}$ if and only if $T = T_\nu$ for some category ν on M.
Proof. If $\nu = \nu_T$, then $\nu_{T_\nu} = \nu_{T_{\nu_T}} = \nu_T$ by the above proposition and hence $\nu_{T_\nu} = \nu$. Similarly, if $T = T_\nu$, then $T_{\nu_T} = T_{\nu_{T_\nu}} = T_\nu = T$. $\square$

5 Cohomological Category

This section concerns two examples: Cohomology Ljusternik-Schnirelmann category and Cup Length category. From now on, H^* is assumed to be a fixed cohomology functor on a topological space M with a cup product $\cup$.

Definition. A subset $A \subset M$ is called cohomologically trivial in M if the restriction map $j^* : H^k(M) \longrightarrow H^k(A)$ is zero for $k > 0$. The set of all cohomologically trivial open subsets of M is denoted by $T_c(M)$. For every subset $A \subset M$, we define the cup length of A (in M) to be the minimum integer $N > 0$ such that for any set of cohomology classes $\alpha_j \in H^{k_j}(M)$, $j = 1, \cdots, N$ of degree $k_j \geq 1$, the class $(\alpha_1 \cup \cdots \cup \alpha_N)|_A = j^*(\alpha_1 \cup \cdots \cup \alpha_N)$ is zero. Hence A is cohomologically trivial if and only if $cuplength(A) = 1$.

Proposition 5.1. Suppose that A and B are excisive in X and $\alpha, \beta \in H^*(X)$ with $\alpha|_A = \beta|_B = 0$, then $(\alpha \cup \beta)|_{A \cup B} = 0$. In particular if $X = U \cup V$ where U and V are open sets and $\alpha|_U = \beta|_V = 0$, then $\alpha \cup \beta = 0$.

Proof. Since A and B are excisive, the cup product map $H^*(X, A) \otimes H^*(X, B) \longrightarrow H^*(X, A \cup B)$ is defined [2] and the following diagram commutes:

$$\begin{array}{ccccc} H^*(X,A) & \otimes & H^*(X,B) & \longrightarrow & H^*(X, A\cup B) \\ \downarrow & & \downarrow & & \downarrow \\ H^*(X) & \otimes & H^*(X) & \longrightarrow & H^*(X) \\ \downarrow & & \downarrow & & \downarrow \\ H^*(A) & \otimes & H^*(B) & & H^*(A \cup B) \end{array}$$

the rest of the proof is a diagram chase and similar to Lemma 4 in [1]. $\square$

Lemma 5.7. Suppose that $f : X \longrightarrow Y$ is a continuous map such $f^* : H^k(Y) \longrightarrow H^k(X)$ is surjective for $k > 0$. Then $cuplength(A) \leq cuplength(f(A))$ for every $A \subseteq X$, .

Proof. We may assume that $m = cuplength(f(A)) < \infty$. Now for $\alpha_i \in H^{k_i}(X), i = 1, \cdots, m$ with $k_i > 0$, there are $\beta_i \in H^{k_i}(Y)$ such that $f^*(\beta_i) = \alpha_i$. Consider the following diagram

$$\begin{tikzcd} H^k(X) \arrow[d, "i^*"] & H^k(Y) \arrow[l, "f^*"'] \arrow[d, "j^*"] \\ H^k(A) & H^k(f(A)) \arrow[l, "(f|_A)^*"'] \end{tikzcd}$$

where i^* and j^* are restriction maps. Now we have $(\alpha_1 \cup \cdots \cup \alpha_m)|_A = i^*(\alpha_1 \cup \cdots \cup \alpha_m) = i^*(f^*(\beta_1) \cup \cdots \cup (f^*(\beta_m)) = i^*(f^*(\beta_1 \cup \cdots \cup \beta_m)) = 0$ and the proof in complete. $\square$

The above two propositions show that the cup length map is a precategory and hence it defines a category which is denoted by ν_{CL}. Moreover if we use the above lemma in the case of $cuplength(A) = 1$, we see that T_c is a T-collection and hence defines a category which is denoted by ν_c. It is easy to see that $T_{\nu_{CL}} = T_c \cup \{\varnothing\}$, hence $\nu_c = \nu_{T_{\nu_{CL}}}$ which is a relation between CLS and CL categories. Moreover by Lemma 4.1, $\nu_{CL} \leq \nu_{T_{\nu_{CL}}} = \nu_c$ and since $T_H \subset T_c$, we have $\nu_{CL} \leq \nu_c \leq \nu_H$. These inequalities show that the best estimate is provided by ν_H, but ν_{CL} is somewhat easier to compute.

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